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Representations of the superalgebra F_4 and Young supertableaux†

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Abstract. We present a method for constructing typical as well as atypical finite-dimensional representations of the superalgebra F_4. For such a purpose, Young supertableaux are introduced.

1. Introduction

Among simple Lie superalgebras [1], the superalgebra F_4 occupies a peculiar position. It is one of the three exceptional superalgebras, and also the only one simple superalgebra with a fermionic part completely spinorial under its bosonic one. Indeed its odd sector transforms as a (2, 8) under the even one $A_1 \times B_3$. As a real form this bosonic part is a non-compact form of $SU(2) \times O(7)$, or more precisely $SU(2) \times Spin(7)$. It might be interesting to remark that the sixteen-dimensional representation of SO(10), considered as a grand unification group, reduces under $SU(2) \times SO(7)$ as $16 = (2, 8)$ and therefore that the fermionic part of F_4 can be directly associated with the 16-plot of quarks and leptons of one family with the correct quantum numbers of colour and electric charge. The relevance of F_4 might also show up in supergravity theories since its orthogonal part can be seen as the de Sitter group in $d = 6$ dimensions, while the Clifford algebra in $d = 6$ is precisely eight dimensional [2, 3].

In this paper we want to consider the finite-dimensional representations of F_4. From the general classification of finite-dimensional representations for simple superalgebras [4], explicit studies have been done in the case of unitary superalgebras [5] and orthosymplectic ones [6] for which Young supertableaux have also been introduced [7-9]. In a way analogous to that used in reference [8] for $OSp(M/N)$ superalgebras, we build up a procedure to decompose a F_4 representation into representations of its bosonic part. We must mention the work of Thierry-Mieg who has been able, using the Weyl symmetry [9], to construct numerically representations of simple superalgebras [10]. Moreover a first attempt to introduce Young tableaux for F_4 has been done in reference [11], but this last study is far from being general.

2. The superalgebra F_4

Finite-dimensional irreducible representations of superalgebras can be characterised by their highest weight in the root space, or equivalently by means of Kac–Dynkin labels [4].

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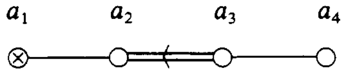
The Kac–Dynkin diagram for the rank-4 superalgebra F_4 is



(2.1)

where a_2, a_3, a_4 are positive or null integers. For the SO(7) part, a_2 is the shorter root and the relation between Dynkin labels (a_2, a_3, a_4) and Young tableau labels $(\lambda_1, \lambda_2, \lambda_3)$ is

$$a_4 = \lambda_1 - \lambda_2 \qquad a_3 = \lambda_2 - \lambda_3 \qquad a_2 = 2\lambda_3. \tag{2.2}$$

In (2.1) the $\mathrm{Sp}(2) \simeq \mathrm{SU}(2)$ representation label is hidden by the odd root $\overset{a_1}{\otimes}$ and its value $(=2j)$ is given by

$$b = \tfrac{1}{3}(2a_1 - 3a_2 - 4a_3 - 2a_4). \tag{2.3}$$

We note that (2.3) implies a_1 to be integer or half-integer.

As the adjoint representation of F_4 decomposes as $(1,2)+(3,1)+(2,8)$ under $\mathrm{SU}(2)\times\mathrm{SO}(7)$, the system of roots is

$$\begin{aligned} &\text{even part} \qquad \Delta_0 = \{\pm\delta;\ \pm\varepsilon_i \pm \varepsilon_j;\ \pm\varepsilon_i\} \qquad i,j = 1,2,3,4 \\ &\text{odd part} \qquad \Delta_1 = \{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta)\}. \end{aligned} \tag{2.4}$$

Denoting $\{h_i\}$ $(i = 1, \ldots, 4)$ a basis of the Cartan subalgebra, we deduce from (2.4) the four simple positive (negative) roots β_1^+, β_j^+ (β_1^-, β_j^-), $j = 2, 3, 4$:

$$\beta_1^+ = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta) \qquad \alpha_2^+ = -\varepsilon_1 \qquad \alpha_3^+ = \varepsilon_1 - \varepsilon_2 \qquad \alpha_4^+ = \varepsilon_2 - \varepsilon_3 \tag{2.5}$$

which satisfy

$$\{\beta_1^+, \beta_1^-\} = h_1 \qquad [\alpha_2^+, \alpha_2^-] = h_2 \qquad [\alpha_3^+, \alpha_3^-] = 2h_3 \qquad [\alpha_4^+, \alpha_4^-] = 2h_4. \tag{2.6}$$

The Cartan matrix, defined as

$$[h_i, \alpha_j^\pm] = \pm a_{ij}\alpha_j^\pm \qquad i,j = 1,2,3,4 \tag{2.7}$$

(in which we identify $\alpha_1 \equiv \beta_1$) then becomes

$$a_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{2.8}$$

Note that for convenience we use the same notation for the root and its corresponding operator.

The different $\mathrm{Sp}(2)\times\mathrm{SO}(7)$ representations contained in the representation (a_1, a_2, a_3, a_4) of F_4 are obtained by repeated application of the odd negative roots β_k^- $(k = 1, \ldots, 8)$ on the highest weight Λ (which satisfies $h_i\Lambda = a_i$, $i = 1, 2, 3, 4$). These eight negative roots form an eight spinorial representation of SO(7) as well as the eight positive odd ones, and can also be obtained from β_1^- as follows [6]:

$$\begin{aligned} &\beta_2^- = [\beta_1^-, \alpha_2^-] \qquad \beta_3^- = [\beta_2^-, \alpha_3^-] \qquad \beta_4^- = [\beta_3^-, \alpha_4^-] \qquad \beta_5^- = [\beta_3^-, \alpha_2^-] \\ &\beta_6^- = [\beta_4^-, \alpha_2^-] = [\beta_5^-, \alpha_4^-] \qquad \beta_7^- = [\beta_6^-, \alpha_3^-] \qquad \beta_8^- = [\beta_7^-, \alpha_2^-] \end{aligned} \tag{2.9}$$

that is

$$
\begin{aligned}
&\beta_1^- = -\tfrac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta) && \beta_2^- = -\tfrac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta) \\
&\beta_3^- = -\tfrac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta) && \beta_4^- = -\tfrac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta) \\
&\beta_5^- = -\tfrac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\delta) && \beta_6^- = -\tfrac{1}{2}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\delta) \\
&\beta_7^- = -\tfrac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\delta) && \beta_8^- = -\tfrac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\delta).
\end{aligned}
\tag{2.10}
$$

One can easily deduce, using (2.6) and (2.9), that the anticommutation relations $\{\beta_i^+, \beta_i^-\}$, $i=1,\ldots,8$, give, up to a possible multiplicative factor, h_1, h_1-h_2, $h_1-h_2-2h_3$, $h_1-h_2-2h_3-2h_4$, $h_1-2h_2-2h_3$, $h_1-2h_2-2h_3-2h_4$, $h_1-2h_2-4h_3-2h_4$, $h_1-3h_2-4h_3-2h_4$.

In the following a $\mathrm{Sp}(2)\times \mathrm{O}(7)$ representation will be said to belong to the ith level, $1\leqslant i\leqslant 8$, if it is obtained by the ith-fold antisymmetric product of i negative fermionic roots on Λ.

The action of β_k^- $(k=1,\ldots,8)$ on $\Lambda=(a_1, a_2, a_3, a_4)$ can be easily deduced using (2.9). In order to use more conveniently the following relations in the next section we develop Λ as follows:

$$\Lambda=(a_1, a_2, a_3, a_4; b; \lambda_1, \lambda_2, \lambda_3). \tag{2.11}$$

Then

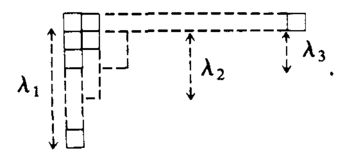
$$
\begin{aligned}
&\beta_1^-\Lambda=(a_1, a_2+1, a_3, a_4; b-1; \lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \lambda_3+\tfrac{1}{2}) \\
&\beta_2^-\Lambda=(a_1-1, a_2-1, a_3+1, a_4; b-1; \lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \lambda_3-\tfrac{1}{2}) \\
&\beta_3^-\Lambda=(a_1-1, a_2+1, a_3-1, a_4+1; b-1; \lambda_1+\tfrac{1}{2}, \lambda_2-\tfrac{1}{2}, \lambda_3+\tfrac{1}{2}) \\
&\beta_4^-\Lambda=(a_1-1, a_2+1, a_3, a_4-1; b-1; \lambda_1-\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \lambda_3+\tfrac{1}{2}) \\
&\beta_5^-\Lambda=(a_1-2, a_2-1, a_3, a_4+1; b-1; \lambda_1+\tfrac{1}{2}, \lambda_2-\tfrac{1}{2}, \lambda_3-\tfrac{1}{2}) \\
&\beta_6^-\Lambda=(a_1-2, a_2-1, a_3+1, a_4-1; b-1; \lambda_1-\tfrac{1}{2}, \lambda_2+\tfrac{1}{2}, \lambda_3-\tfrac{1}{2}) \\
&\beta_7^-\Lambda=(a_1-2, a_2+1, a_3-1, a_4; b-1; \lambda_1-\tfrac{1}{2}, \lambda_2-\tfrac{1}{2}, \lambda_3+\tfrac{1}{2}) \\
&\beta_8^-\Lambda=(a_1-3, a_2-1, a_3, a_4; b-1; \lambda_1-\tfrac{1}{2}, \lambda_2-\tfrac{1}{2}, \lambda_3-\tfrac{1}{2}).
\end{aligned}
\tag{2.12}
$$

Finally, we note that a F_4 representation with $b<4$ has to satisfy a consistency condition, i.e.

$$
\begin{aligned}
&b=0 && a_i=0 \qquad i=2,3,4 \\
&b=1 && \text{not possible} \\
&b=2 && a_2=a_4=0 \\
&b=3 && a_2=2a_4+1.
\end{aligned}
\tag{2.13}
$$

3. The content of a F_4 representation and Young supertableaux

To the F_4 representation labelled by $(b; a_2, a_3, a_4)$ can be associated a Young supertableau (YST) defined as follows: (i) its first row contains b boxes and (ii) the Young tableau (YT) obtained after erasing the first row is just the transpose of the YT corresponding to the representation (a_2, a_3, a_4) of $\mathrm{O}(7)$:

 (3.1)

If a_2 is odd, this O(7) Young tableau will be related to a spinorial O(7) representation. Using the convention of reference [8] each box of its first column will contain the letter 's': $\boxed{s}$ representing 'half a box'. It follows that in the case of O(7) spinor representations the corresponding YST of F_4 contain in the second row three $\boxed{s}$ boxes while the lower rows contain (if any) usual boxes $\square$. Such a framework has been developed by the authors for computing products of orthogonal group representations (see the appendix of reference [8]) and references therein). In case of F_4, we will call spinorial a representation $(b; a_2, a_3, a_4)$ with b odd and a_2 even, or b even and a_2 odd, and vectorial the other F_4 representations. We remark that the drawing of a graphically meaningful YST according to the rules for usual YT implies automatically the consistency relations (2.13).

In the following we give rules in order to obtain from a F_4 YST the content of the corresponding representation in terms of representations of the bosonic algebra $SU(2) \times O(7)$. We have to distinguish typical representations from atypical ones.

3.1. *Typical representations*

A necessary (but not sufficient!) condition for a representation to be typical is that $b \geqslant 4$.

For $b \geqslant 7$, the decomposition formula for a typical representation $(b, [\lambda])$ into $SU(2) \times O(7)$ representation reads, in terms of Young tableaux,

$$(b,[\lambda]) = \sum_{k=0}^{\min(b,8)} \Bigg(\underbrace{\square\square\cdots\square}_{b-k}, \Bigg\{ \underbrace{\begin{matrix}\boxed{S}\\ \boxed{S}\\ \boxed{S}\end{matrix} \times \cdots \times \begin{matrix}\boxed{S}\\ \boxed{S}\\ \boxed{S}\end{matrix}}_{k} \Bigg\}_{A} \times [\lambda] \Bigg) \tag{3.2}$$

where in the RHS of (3.2) the first row Young tableau is relative to SU(2) representations and the other ones to O(7) representations. The symbol $\times$ stands for the Kronecker product and the subscript A for 'antisymmetric'.

The justification of (3.2) is in the property of any F_4 irreducible representation to appear as a sum of $SU(2) \times O(7)$ representations obtained from the highest weight Λ by the repeated application of the negative fermionic generators which belong to the spinorial eight-dimensional representation $(\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2})$ of O(7). One can see from (2.12) that by action of a negative fermionic generator, b decreases by one unit. Moreover, from the (anti-) commutation relations given in (2.6) and (2.7) one deduces that only antisymmetric combinations of negative fermionic generators allow us to reach the different $SU(2) \times O(7)$ representations. The O(7) representations showing up at the kth level will then be obtained from the Kronecker product of $[\lambda]$ by the antisymmetric (A subscript) k-times product of the fundamental spinorial O(7) representation. We list here the O(7) representations which appear in the antisymmetric

k-fold product $(\mathbf{8}\times\ldots\times\mathbf{8})_A$:

$$\begin{array}{ll} k=1 & [\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]=\mathbf{8} \\ k=2 & [1,1,0]\oplus[1,0,0]=\mathbf{21}\oplus\mathbf{7} \\ k=3 & [\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}]\oplus[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]=\mathbf{48}\oplus\mathbf{8} \\ k=4 & [1,1,1]\oplus[2,0,0]\oplus[1,0,0]\oplus[0,0,0]=\mathbf{35}\oplus\mathbf{27}\oplus\mathbf{7}\oplus\mathbf{1} \\ k>4 & \text{same result as for } (8-k). \end{array} \tag{3.3}$$

For $7>b\geqslant 4$, (3.2) has to be slightly modified since not all the O(7) representations constituting the k-fold antisymmetric product of $[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]$ by itself are present. We will then have to consider

$$\begin{array}{lll} b=4 & k=1 & [\lambda]\times[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}] \\ & k=2 & [\lambda]\times\{[1,1,0]\oplus[1,0,0]-[0,0,0]\} \\ & k=3 & [\lambda]\times[\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}] \\ & k=4 & [\lambda]\times\{[1,1,1]\oplus[2,0,0]\oplus[0,0,0]-[1,1,0]\} \end{array} \tag{3.4}$$

$$\begin{array}{lll} b=5 & k=1,2,3 & \text{as in (3.3)} \\ & k=4 & [\lambda]\times\{[1,1,1]\oplus[2,0,0]\oplus[1,0,0]\} \\ & k=5 & [\lambda]\times[\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}] \end{array} \tag{3.5}$$

$$\begin{array}{lll} b=6 & k=1,2,3,4,5 & \text{as in (3.3)} \\ & k=6 & [\lambda]\times\{[1,1,0]\oplus[1,0,0]-[0,0,0]\}. \end{array} \tag{3.6}$$

The reason for these modifications is due to the fact that for $b<8$, the labels of the highest weight Λ have to be such that the k-fold, $k>b$, product of ordered fermionic generators have to give a decoupled state. Taking as an example the case $b=4$, one can check that subtracting the trivial representation $[0,0,0]$ at level $k=2$ implies the subtraction of $[0,0,0]\times[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]$, i.e. of the $\mathbf{8}$ representation at $k=3$, then of the $\mathbf{8}\times\mathbf{8}|_A=\mathbf{21}\oplus\mathbf{7}$ representations at $k=4$, and finally of the $\mathbf{48}\oplus\mathbf{8}$ representations at $k=5$. As the antisymmetric product of five times the representation $\mathbf{8}$ by itself gives, as for $k=3$, the representations $\mathbf{48}\oplus\mathbf{8}$, the $k=(b+1)$th level is then automatically empty.

The above method allows one, of course, to reconstruct the formula providing the dimension of a F_4 typical representation

$$\dim(b;[\lambda])=2^8(b-3)\dim[\lambda]. \tag{3.7}$$

3.2. *Atypical representations*

The property for a representation to be atypical can be expressed in different ways [1]. One can say that a finite-dimensional representation of a simple superalgebra with highest weight Λ is atypical if there exists a positive fermionic root α, with 2α not being an even root and satisfying

$$(\Lambda+\rho,\alpha)=0 \qquad \text{with } \rho=\rho_0-\rho_1 \tag{3.8}$$

where ρ_0 (respectively ρ_1) is the half-sum of all the even (respectively odd) positive simple roots. In order to formulate this property in a more concrete way, let us consider a F_4 representation labelled by $(a_1, \ldots, a_4)$ and such that $(\Lambda+\rho_1 \beta_1^+)=0$, i.e. $a_1=0$. One can notice that, Λ being the highest weight, $\beta_1^+\beta_1^-\Lambda=\{\beta_1^-, \beta_1^+\}\Lambda=h_1\Lambda=a_1\Lambda$ and therefore $\beta_1^+\beta_1^-\Lambda=0$. The condition $a_1=0$ will be called the first atypical condition. Note that for convenience, we have used the same notation for the root and its corresponding operator.

There will be eight possible atypicality conditions for F_4. We recall that only antisymmetrised products of negative fermionic roots will allow one to go down the different 'floors' of a superalgebra representation, the symmetric combinations belonging to the bosonic algebra part. Actually these eight conditions will correspond to the relations

$$(\Lambda+\rho, \beta_i^+)=0 \qquad i=1, \ldots, 8 \tag{3.9}$$

or to

$$X_i^+ X_i^- \Lambda=0 \tag{3.10}$$

with $X_i^-=\beta_1^-$; $\beta_2^-\beta_1^-$; $\beta_3^-\beta_2^-\beta_1^-$; $\beta_4^-\beta_3^-\beta_2^-\beta_1^-$; $\beta_5^-\beta_3^-\beta_2^-\beta_1^-$; $\beta_6^-\beta_3^-\beta_2^-\beta_1^-$; $\beta_7^-\beta_6^-\beta_3^-\beta_2^-\beta_1^-$; $\beta_8^-\beta_7^-\beta_6^-\beta_3^-\beta_2^-\beta_1^-$.

We note that the operators X_i^-, $8 \geqslant i \geqslant 5$, are not the product of i negative fermionic roots as it appears for the other kinds of simple superalgebras: this is a feature of the F_4 superalgebra in which the fermionic roots belong to the spinorial fundamental representation of its bosonic part O(7) and not to the vector fundamental representation. We list here these eight atypicality conditions [1]:

$$\begin{array}{llll}
(1) & a_1=0 & \text{or} & b=0 \\
(2) & a_1=a_2+1 & \text{or} & b=\tfrac{1}{3}(2-a_2-4a_3-2a_4) \\
(3) & a_1=a_2+2a_3+3 & \text{or} & b=\tfrac{1}{3}(6-a_2-2a_4) \\
(4) & a_1=a_2+2a_3+2a_4+5 & \text{or} & b=\tfrac{1}{3}(10-a_2+2a_4) \\
(4') & a_1=2a_2+2a_3+4 & \text{or} & b=\tfrac{1}{3}(8+a_2-2a_4) \\
(5) & a_1=2a_2+2a_3+2a_4+6 & \text{or} & b=\tfrac{1}{3}(12+a_2+2a_4) \\
(6) & a_1=2a_2+4a_3+2a_4+8 & \text{or} & b=\tfrac{1}{3}(16+a_2+4a_3+2a_4) \\
(7) & a_1=3a_2+4a_3+2a_4+9 & \text{or} & b=\tfrac{1}{3}(18+3a_2+4a_3+2a_4).
\end{array} \tag{3.11}$$

In the construction of an atypical representation, we have to decouple an invariant subspace. This can be done as follows. Let Λ be an atypical representation of type (i). We start by making a decomposition using (3.2) for any value of the non-negative integer b. Let V be the set of $\mathrm{Sp}(2)\times\mathrm{O}(7)$ representations obtained by this method. If at the first level, a $\mathrm{Sp}(2)\times\mathrm{O}(7)$ highest weight Λ' satisfying the same ith atypical condition does appear, Λ' has to be considered as the highest weight of the atypical subspace. Let V_0 be the decomposition of the F_4 representation associated with Λ'. Again using (3.2), the atypical representation identified by Λ will be obtained by taking away from V all the $\mathrm{Sp}(2)\times\mathrm{O}(7)$ representations appearing in V_0. If there is no such Λ', we then have to look for a $\mathrm{Sp}(2)\times\mathrm{O}(7)$ highest weight Λ'' satisfying the $(i-1)$th atypical condition at the second level of the decomposition of Λ. Denoting by V_0' the decomposition of Λ'' into $\mathrm{Sp}(2)\times\mathrm{O}(7)$ representations, we will then have to take away from V the $\mathrm{Sp}(2)\times\mathrm{O}(7)$ representations present in V_0'. If no such Λ'' exists, we must look for a highest weight Λ''' satisfying the $(i-2)$th atypical condition in the third level of Λ decomposition and proceed as before, and so on.

The justification of the above statements lies in the following observations.

(i) If $(\Lambda+\rho, \beta_i^+)=0$ then $(\Lambda'+\rho, \beta_i^+)=0$ with $\Lambda'=\Lambda-\beta_i^+$.

(ii) If Λ' does not appear in V, i.e. Λ' is characterised by Dynkin labels which do not specify a highest weight of a $\mathrm{Sp}(2)\times\mathrm{O}(7)$ irreducible representation, then $(\Lambda''+\rho, \beta_{i-1}^+)=0$ with $\Lambda''=\Lambda'-\beta_{i-1}^+$.

(iii) If $(\Lambda+\rho, \beta_i^+)=0$ and $|\Lambda-\beta_i^+\rangle\in V$ then $\beta_i^+|\Lambda-\beta_i^+\rangle=0$; it follows that $|\Lambda-\beta_i^+\rangle$ behaves as the highest weight of the invariant subspace to decouple.

Before presenting an example of decomposition, let us mention that a formula for the dimension of representations satisfying the 3rd atypicality condition can be obtained as a byproduct of our method. Using (2.11), (2.3) and the property of b to be integer such representations are of the form $(a_1=2a+3,\ a_2=0,\ a_3=a,\ a_4=0)$. With respect to the parameter $a_3=a$, their dimensions are

$$
\begin{aligned}
\dim(2a+3, 0, a, 0) &= \tfrac{1}{360}(a+1)(a+2)(a+3)(2a+3)(2a+5)\\
&\quad\times\left(6(a^2+4a+2)+\frac{8(a+2)(a+3)(a+4)}{(2a+3)}+\frac{8a(a+1)(a+2)}{(2a+5)}\right.\\
&\quad\left.+\frac{(a+3)^2(a+4)(2a+7)}{(a+1)(2a+3)}+\frac{a(a+1)^2(2a+1)}{(a+3)(2a+5)}\right).
\end{aligned}
\tag{3.12}
$$

4. Example

Let us illustrate our method by decomposing the following F_4 representation:

$$(a_1=8,\ a_2=a_3=1,\ a_4=0)\equiv(b=3;\ [\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}])$$

which verifies the 4th and 4'th atypical condition. The decomposition of this representation, as if it were *typical*, is

$$
\begin{aligned}
&(0) && (\underset{\mathbf{4}}{3};\ \underset{\mathbf{112}}{[\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]})\\
&(1) && (2;\ [\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]\times[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}])\\
& && \quad=(\underset{\mathbf{3}}{2};\ \underset{\mathbf{378}}{[221]}+\underset{\mathbf{189}}{[211]}+\underset{\mathbf{168}}{[22]}+\underset{\mathbf{105}}{[21]}+\underset{\mathbf{35}}{[111]}+\underset{\mathbf{21}}{[11]})\\
&(2) && (1;\ [\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]\times\{[11]\times[1]\})\\
& && \quad=(\underset{\mathbf{2}}{1};\ \underset{\mathbf{720}}{[\tfrac{5}{2},\tfrac{5}{2},\tfrac{1}{2}]}+\underset{\mathbf{560}}{[\tfrac{5}{2},\tfrac{3}{2},\tfrac{3}{2}]}+2\underset{\mathbf{112}}{[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}]}+2\underset{\mathbf{512}}{[\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2}]}\\
& && \qquad+\underset{\mathbf{168}}{[\tfrac{5}{2},\tfrac{1}{2},\tfrac{1}{2}]}+3\underset{\mathbf{112}}{[\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]}+2\underset{\mathbf{48}}{[\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}]}+\underset{\mathbf{8}}{[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]})\\
&(3) && (0;\ [\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]\times\{[\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}]+[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\})\\
& && \quad=(\underset{\mathbf{1}}{0};\ \underset{\mathbf{1617}}{[321]}+\underset{\mathbf{294}}{[222]}+\underset{\mathbf{693}}{[32]}+\underset{\mathbf{616}}{[311]}+3\underset{\mathbf{378}}{[221]}+\underset{\mathbf{330}}{[31]}\\
& && \qquad+2\underset{\mathbf{168}}{[22]}+4\underset{\mathbf{189}}{[211]}+3\underset{\mathbf{105}}{[21]}+3\underset{\mathbf{35}}{[111]}+\underset{\mathbf{27}}{[2]}+2\underset{\mathbf{21}}{[11]}+\underset{\mathbf{7}}{[1]}).
\end{aligned}
\tag{4.1}
$$

At first level, the representation (2; [2, 1]) appears, which satisfies the 4'th atypical condition; its highest state has to be considered as the *highest weight* of an invariant subspace to be decoupled. Its decomposition as if it were *typical* is

$$\begin{array}{ll}(0) & (2;[21]) \\ (1) & (1;[\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2}]+[\tfrac{5}{2},\tfrac{1}{2},\tfrac{1}{2}]+[\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]+[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]) \\ (2) & (0;[32]+[311]+[221]+[211]+[3]+2[21]+[111]+[1]).\end{array} \tag{4.2}$$

At second level, the representation $(1;[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}])$ appears, which satisfies the 3rd atypical condition, and its highest state has to be considered as the highest weight of an invariant subspace to be decoupled. Its decomposition as if it were *typical* is

$$\begin{array}{ll}(0) & (1;[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}]) \\ (1) & (0;[222]+[221]+[211]+[111]).\end{array} \tag{4.3}$$

Drawing away from (4.1) the $Sp(2)\times O(7)$ representations which appear in (4.2) and (4.3), we get the correct decomposition of the *atypical* representation $(3;[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}])$, i.e.

$$\begin{array}{ll}(0) & (3;[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}]) \\ (1) & (2;[221]+[211]+[22]+[111]+[11]) \\ (2) & (1;[\tfrac{5}{2},\tfrac{5}{2},\tfrac{1}{2}]+[\tfrac{5}{2},\tfrac{3}{2},\tfrac{3}{2}]+[\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2}]+[\tfrac{3}{2},\tfrac{3}{2},\tfrac{3}{2}] \\ & \quad +2[\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2}]+[\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}]+[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]) \\ (3) & (0;[321]+[221]+[211]+[21]+[111]+[11]).\end{array} \tag{4.4}$$

The dimension of the representation is 9702, in agreement with reference [10].

References

[1] Kac V 1977 *Adv. Math.* **26** 8; 1977 *Commun. Math. Phys.* **53** 33
[2] De Witt B S and van Nieuwenhuizen P 1982 *J. Math. Phys.* **23** 1953
[3] Giani P, Pernici M and van Nieuwenhuizen P 1984 *Preprint ITP-SB-84-35*
[4] Kac V 1978 *Lecture Notes in Mathematics* **676** (Berlin: Springer)
[5] Hurni J P and Morel B 1983 *J. Math. Phys.* **24** 157
[6] Hurni J P and Morel B 1982 *J. Math. Phys.* **23** 2236
[7] Balantekin B and Bars I 1981 *J. Math. Phys.* **22** 1149, 1810
King R C 1982 *Proc. 11th Int. Conf. on Group Theory, Istanbul*
Chen J-Q, Chen X-G and Gao M-J 1983 *J. Phys. A: Math. Gen.* **16** 1361
Bars I, Morel B and Ruegg H 1983 *J. Math. Phys.* **24** 2253
Delduc F and Gourdin M 1984 *J. Math. Phys.* **25** 1651; 1985 *J. Math. Phys.* **26** 1865
[8] Morel B, Sciarrino A and Sorba P 1985 *J. Phys. A: Math. Gen.* **18** 1597
[9] Thierry-Mieg J 1984 *Proc. 13th Int. Colloq. on Group Theoretical Methods in Physics* (*Lecture Notes in Physics* **201**) (Berlin: Springer) p 94
[10] Thierry-Mieg J *Table des Représentations Irréductibles des Superalgèbres de Lie* unpublished
[11] Hurni J P 1984 *Preprint, UGVA-DPT 1984/04-426* Université de Genève